

Computing Super Matrix Invariants

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Abstract

In [5] we generalized the first and second fundamental theorems of invariant theory from the general linear group to the general linear Lie superalgebra. In the current paper we generalize the computations of the numerical invariants (multiplicities and Poincaré series) to the superalgebra case. The results involve either inner products of symmetric functions in two sets of variables, or complex integrals.

Keywords: general linear Lie superalgebra, generic trace rings, hook Schur functions, invariants, trace identities,

Introduction

Formanek's paper [7], based partly on Procesi's [9], has been a major influence on my work. One of the things Formanek does is describe a certain question which can be posed from five different points of view: (1) Inner products of symmetric group characters; (2) plethysms of symmetric functions; (3) invariants of matrices; (4) trace rings of generic matrices; and (5) complex integrals. Here is a brief description of each:

Problem 1 Let $\Lambda_k(n)$ be the partitions of n into at most k parts, let $\Lambda_k = \cup_n \Lambda_k(n)$, and let χ^λ denote the character of the symmetric group S_n on the partition λ . Then the sum $\sum_{\lambda \in \Lambda_k(n)} \chi^\lambda \otimes \chi^\lambda$ decomposes into irreducible characters as $\sum m_\lambda \chi^\lambda$ and we would like to evaluate the multiplicities m_λ .

Problem 2 Let S_λ denote the Schur function on the partition λ . Let X denote the set of variables $\{x_1, \dots, x_k\}$, and let $S_\lambda(XX^{-1})$ denote S_λ evaluated on all xy^{-1} , $x, y \in X$ (including n 1's). Then m_λ also equals $\langle S_\lambda(XX^{-1}), 1 \rangle$, where the inner product is the natural inner product on symmetric functions as in I.4 of [8].

Problem 3 Consider functions $\phi : M_k(F)^n \rightarrow F$ which are polynomial in the entries and such that

$$\phi(gA_1g^{-1}, \dots, gA_ng^{-1}) = \phi(A_1, \dots, A_n)$$

for all $A_1, \dots, A_n \in M_k(F)$ and all $g \in GL_k(F)$. Such a function is said to be invariant under conjugation from $GL_k(F)$. They form a ring with an n -fold grading and determine a Poincaré series $P(k, n)$. Since $P(k, n)$ is a symmetric function it can be expanded into Schur functions as

$$P(k, n) = \sum m_\lambda S_\lambda(t_1, \dots, t_n)$$

where the m_λ are the heros of Problems 1 and 2.

Problem 4 Let X_α be the generic $k \times k$ matrix with entries $X_\alpha = (x_{ij}^{(\alpha)})$, and let $R(k, n)$ be the algebra generated by X_1, \dots, X_n . Let $\bar{C}(k, n)$ be the commutative algebra generated by traces of elements of $R(k, n)$. Then $\bar{C}(k, n)$ has an n -fold grading and so a Poincaré series in n variables. This series equals $P(k, n)$ from Problem 3.

Problem 5 The function $P(k, n)$ can be evaluated as the following complex integral:

$$(2\pi i)^{-k} (k!)^{-1} \oint_T \frac{\prod_{i \neq j} (1 - z_i z_j^{-1})}{\prod_{i,j} \prod_{\alpha} (1 - z_i z_j^{-1} t_{\alpha})} \frac{dz_1}{z_1} \wedge \cdots \wedge \frac{dz_k}{z_k}$$

where $i, j = 1, \dots, k$ and $\alpha = 1, \dots, n$, and where T is the torus $|z_i| = 1$.

Having five ways to look at the same object is useful for proving things about it. Properties 3 and 5 are useful for actual computations, but also have theoretical consequences. Using (3) it is easy to show that $m_{\lambda} = 0$ if $\lambda \notin \Lambda_{k^2}$, and if $\lambda \in \Lambda_{k^2}$ and $\mu = (\lambda_1 + a, \dots, \lambda_{k^2} + a)$, then $m_{\lambda} = m_{\mu}$. Using (5) one can show that $P(n, k)$ is a rational function, that it can be written with denominator a product of terms of the form $(1 - u)$, where u is a monic monomial of degree at most k , and that $P(n, k)$ satisfies the functional equation

$$P(t_1^{-1}, \dots, t_n^{-1}) = (-1)^g (t_1 \cdots t_n)^{k^2} P(t_1, \dots, t_k), \quad (1)$$

where $g = (n - 1)k^2 + 1$.

This theory has a \mathbb{Z}_2 -graded analogue which we sketch briefly, see [2]. In this theory there are analogues of (1), (3) and (4), but not of (2) and (5). Here are the analogues:

Problem 1a Let $H(k, \ell; n)$ be the set of partitions of n in which at most k parts are greater than or equal to ℓ , and let $H(k, \ell) = \cup_n H(k, \ell; n)$. Using the standard notation for partitions $\lambda = (\lambda_1, \lambda_2, \dots)$ with $\lambda_1 \geq \lambda_2 \geq \dots$, we have

$$\lambda \in H(k, \ell) \iff \lambda_{k+1} \leq \ell.$$

The character of interest is $\sum \chi^{\lambda} \otimes \chi^{\lambda}$ summed over $\lambda \in H(k, \ell; n)$ and we define the multiplicities m_{λ} by

$$\sum_{\lambda \in H(k, \ell; n)} \chi^{\lambda} \otimes \chi^{\lambda} = \sum m_{\lambda} \chi^{\lambda}.$$

Problem 3a Let E be an infinite dimensional Grassmann algebra. It has a natural \mathbb{Z}_2 grading. We give the set $\{1, \dots, k + \ell\}$ a \mathbb{Z}_2 grading via

$$\deg(i) = \begin{cases} \bar{0} & \text{if } 1 \leq i \leq k \\ \bar{1} & \text{if } k + 1 \leq i \leq k + \ell \end{cases}$$

and then grade the pairs $\{(i, j)\}_{i,j=1}^{k+\ell}$ via $\deg(i, j) = \deg(i) + \deg(j)$.

The algebra $M_{k,\ell}$ is a subalgebra of $M_{k+\ell}(E)$ defined as the set of matrices (a_{ij}) such that for each (i, j) the entry $a_{ij} \in E$ is homogeneous and has the same \mathbb{Z}_2 degree as (i, j) . Then $M_{k,\ell}$ is an algebra and it has a non-degenerate trace with values in E_0 given by

$$\text{tr}(a_{ij}) = \sum (-1)^{\deg(i)} a_{ii}.$$

The group of units of $M_{k,\ell}$ is denoted $PL(k, \ell)$ and is called the general linear Lie superalgebra.

Finally, we consider functions $\phi : M_{k,\ell}^n \rightarrow E$, polynomial in the entries and invariant under conjugation from $PL(k, \ell)$. These form an n -fold graded algebra with Poincaré series $\sum m_\lambda S_\lambda(t_1, \dots, t_n)$, the same m_λ as in problem 1a.

Problem 4a Let $x_{ij}^{(\alpha)}$ be commuting indeterminants and let $e_{ij}^{(\alpha)}$ be anti-commuting indeterminants, so that the algebra $S = F[x_{ij}^{(\alpha)}, e_{ij}^{(\alpha)}]$ will be a free supercommutative algebra. The generic matrix A_α will be the $(k + \ell) \times (k + \ell)$ matrix with (i, j) entry equal to $x_{ij}^{(\alpha)}$ or $e_{ij}^{(\alpha)}$, depending on whether $\deg(i, j)$ equals $\bar{0}$ or $\bar{1}$, respectively. Then the algebra $F[A_1, \dots, A_n]$ will be the generic algebra for $M_{k,\ell}$. It has a trace function with image in S , and we let $\bar{C}(k, \ell; n)$ be the algebra generated by the image of the trace map. This ring has an n -fold grading and a Poincaré series in n variables, the same series $\sum m_\lambda S_\lambda$ from 3a.

It is useful to push this last construction one step farther. Let B_α be the $(k + \ell) \times (k + \ell)$ matrix with (i, j) equal to $e_{ij}^{(\alpha)}$ if $\deg(i, j)$ is $\bar{0}$ and $x_{ij}^{(\alpha)}$ if $\deg(i, j) = \bar{1}$, the opposite of the definition of A_α . Let $R(k, \ell; n, m)$ be the algebra generated by A_1, \dots, A_n and B_1, \dots, B_m . (For the reader familiar with the theory of magnums from [1], this is the magnum of $M_{k,\ell}$.) It has a supertrace function to S and we let $\bar{C}(k, \ell; n, m)$ be the algebra generated by the traces. $\bar{C}(k, \ell; n, m)$ has a $(k + \ell) \times (k + \ell)$ fold grading and a Poincaré series which can be expressed in terms of hook Schur functions as

$$P(k, \ell; n, m) = \sum m_\lambda H S_\lambda(t_1, \dots, t_n; u_1, \dots, u_m),$$

where the m_λ are as in 1a. See [5] for the theory of hook Schur functions.

Such a construction would also be possible in the matrix case (see [4]), but it would be less useful. A basic property of Schur functions is that

$S_\lambda(t_1, \dots, t_k)$ is non-zero precisely when $\lambda \in \Lambda_k$. And, in the matrix case, $m_\lambda = 0$ if $\lambda \notin \Lambda_{k^2}$. This means that we can reconstruct all the m_λ from the Poincaré series $P(k, n)$ as long as $n \geq k^2$. In the case of $M_{k,\ell}$, it is known that $HS_\lambda(x_1, \dots, x_n; y_1, \dots, y_m)$ is non-zero if and only if $\lambda \in H(n, m)$ and that $m_\lambda \neq 0$ only if $\lambda \in H(k^2 + \ell^2; 2k\ell)$. It follows that we get full information about the non-zero m_λ if we know the Poincaré series $H(k, \ell; n, m)$ for some $n \geq k^2 + \ell^2$, $m \geq 2k\ell$.

Our main goal in this paper is to present partial generalizations of problems 2 and 5 to the graded case. Let X denote the set of k variables $\{x_1, \dots, x_k\}$ and let Y denote the set of ℓ variables $\{y_1, \dots, y_\ell\}$. Then for certain λ which we call “large” and which include most of $H(k^2 + \ell^2; 2k\ell)$ we prove

$$m_\lambda = \left\langle \prod_{ij} (1 + x_i y_j^{-1})^{-1} (1 + x_i y_j^{-1})^{-1} HS_\lambda(XX^{-1}, YY^{-1}; XY^{-1}, YX^{-1}), 1 \right\rangle \quad (2)$$

where the inner product is the inner product on functions symmetric on two sets of variables. We will define it explicitly in section 1.

Equation (2) has an application in the case of typical λ . A partition in $H(a; b)$ but not in any strictly smaller hook is called typical, and the set of such is denoted $H'(a; b)$. Such a partition can be thought of as being made up of three parts: The $a \times b$ rectangle, a partition $\alpha(\lambda) \in \Lambda_a$ to the right of the rectangle, and a partition $\beta(\lambda) \in \Lambda_b$ whose conjugate lies below the rectangle. Hopefully Figure 1 makes this clear, but if not we add that if $\lambda = (\lambda_1, \lambda_2, \dots)$ then $\alpha(\lambda) = (\lambda_1 - b, \dots, \lambda_a - b)$ and $\beta(\lambda)$ is the conjugate of $(\lambda_{a+1}, \lambda_{a+2}, \dots)$. The importance of typical partitions for our purposes lies in this factorization theorem for hook Schur functions from [5]. Note that the number of x ’s and y ’s in the theorem equal the dimensions of the hook.

Theorem 0.1 (The Factorization Theorem). *If $\lambda \in H'(a, b)$ with $\alpha = \alpha(\lambda)$ and $\beta = \beta(\lambda)$ then*

$$HS_\lambda(x_1, \dots, x_a; y_1, \dots, y_b) = \prod (x_i + y_j) S_\alpha(x_1, \dots, x_a) S_\beta(y_1, \dots, y_b)$$

Combining the factorization theorem with (2) we get the following.

Theorem 0.2. *Given $\lambda, \mu \in H'(k^2 + \ell^2; 2k\ell)$ with $\alpha(\lambda) = \alpha(\mu) + (a^{k^2 + \ell^2})$ for some a and $\beta(\lambda) = \beta(\mu) + (b^{2k\ell})$ for some b , then $m_\lambda = m_\mu$. (See Figure 2).*

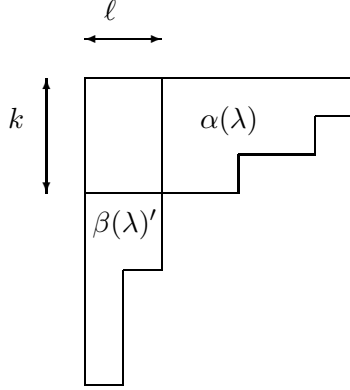


Figure 1: Definition of $\alpha(\lambda)$ and $\beta(\lambda)$

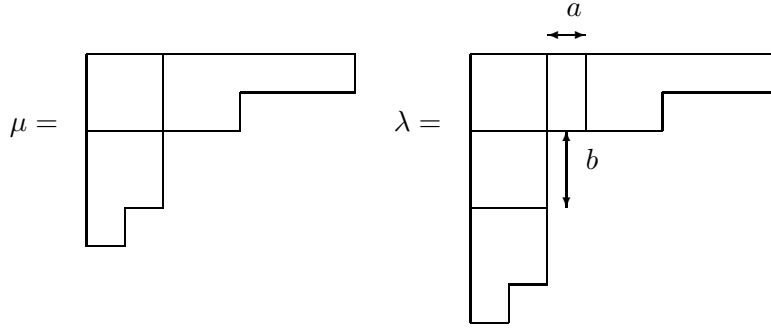


Figure 2: Theorem 0.2

Proof. $S_\alpha(\lambda)(XX^{-1}, YY^{-1}) =$

$$\left(\prod_{z \in XX^{-1}, YY^{-1}} z \right)^a S_{\alpha(\mu)}(XX^{-1}, YY^{-1}) \\ 1^a \cdot S_{\alpha(\mu)}(XX^{-1}, YY^{-1}).$$

And by the same token $S_{\beta(\lambda)}(XY^{-1}, YX^{-1}) = S_{\beta(\mu)}(XY^{-1}, YX^{-1})$. \square

Turning to Poincaré series, since we do not know all of the m_λ we cannot hope to capture the full $\sum m_\lambda S_\lambda$ or $\sum m_\lambda H S_\lambda$, even in small numbers of variables. There are two related infinite series we can express as integrals and derive information about. For the first, let m'_λ be the right hand side

of (2), so $m_\lambda = m'_\lambda$ for large λ . Then we may define

$$P'(k, \ell; a, b) = \sum m'_\lambda H S_\lambda(t_1, \dots, t_a; u_1, \dots, u_b)$$

summed over all $\lambda \in H(a, b)$. Or instead, we may restrict to typical partitions and define

$$T(k, \ell; a, b) = \sum_{\lambda \text{ typical}} m_\lambda S_{\alpha(\lambda)}(t_1, \dots, t_a) S_{\beta(\lambda)}(y_1, \dots, y_b).$$

Using (2) we can write each of these series as a complex integral over a torus. The integrals are too complex to embellish an introduction, and perhaps too complex for much actual computation. However, at least in the case of T we can use the integral to prove that the $T(k, \ell; a, b)$ is the Taylor series of a rational function, we can describe what type of terms occur in the denominator, and we can prove a functional equation similar to (1).

Ideally we would like information about all of the m_λ , not just for large or typical λ . If

$$P(k, \ell; a, b) = \sum m_\lambda H S_\lambda(t_1, \dots, t_a; u_1, \dots, u_b),$$

then it is still open whether $P(k, \ell; a, b)$ is a rational function, what its denominator looks like if it is, and whether it satisfies a functional equation along the lines of (1). See Corollary 4.3 for a case in which it does not.

In the classical case described by Formanek one is also interested in the character

$$\left(\sum_{\lambda \in \Lambda_k(n+1)} \chi^\lambda \otimes \chi^\lambda \right) \downarrow = \sum \bar{m}_\lambda \chi^\lambda,$$

where the arrow indicates inducing down from S_{n+1} to S_n . Each of problems 1 through 5 have analogues in this case. The analogue of the invariant theory problem would concern the invariant maps $M_k(F)^n \rightarrow M_k(F)$. Generalizing to the \mathbb{Z}_2 -graded case we would define

$$\left(\sum_{\lambda \in H(k, \ell; n+1)} \chi^\lambda \otimes \chi^\lambda \right) \downarrow = \sum \bar{m}_\lambda \chi^\lambda$$

and study the \bar{m}_λ . It turns out that analogues of problems 3 and 4 are known in this case, see [2], and that we can now develop analogues of problems 2 and 5, just like we did for m_λ . The theory is very similar. The formula for \bar{m}_λ for large enough λ is the same as equation 2 with an extra factor of $\sum x_i x_j^{-1} + \sum y_i y_j^{-1}$ and the same factor multiplies the integrands in the formulas for the power series $\bar{T}(k, \ell; a, b)$ and $\bar{P}(k, \ell; a, b)$.

1 Computation of Multiplicities

Notation 1.1. Given partitions $\mu, \nu \vdash n$ of the same n , we define the coefficients $\gamma_{\mu, \nu}^\lambda$ via the equation $\chi^\mu \otimes \chi^\nu = \sum_{\lambda \vdash n} \gamma_{\mu, \nu}^\lambda \chi^\lambda$. Note that from Problem 1a this implies

$$m_\lambda = \sum \{\gamma_{\mu, \mu}^\lambda | \mu \in H(k, \ell)\}. \quad (3)$$

We set the stage for the computation of m_λ by quoting two theorems

Theorem 1.2 (Berele-Regev [5]). *If $\mu \in H(k_1, \ell_1)$ and $\nu \in H(k_2, \ell_2)$, then $\gamma_{\mu, \nu}^\lambda = 0$ unless $\lambda \in H(k_1 k_2 + \ell_1 \ell_2, k_1 \ell_2 + \ell_1 k_2)$*

Definition 1.3. Given k, ℓ , we say that a partition λ is large if $\lambda \in H(k^2 + \ell^2; 2k\ell)$ but $\lambda \notin H(a^2 + b^2; 2ab)$ for any $a \leq k, b \leq \ell$ and at least one of the inequalities strict. Note that typical partitions are all large.

Using this definition we get this corollary of Theorem 1.2

Lemma 1.4. *If λ is large and $\mu, \nu \in H(k, \ell)$, then $\gamma_{\mu, \nu}^\lambda \neq 0$ only if μ and ν are typical.*

The next theorem we need is due to Rosas from [12]. It generalizes the classical result that given two sets of variables X and Y , $S_\lambda(XY) = \sum \gamma_{\mu, \nu}^\lambda S_\mu(X) S_\nu(Y)$. This paper of Rosas was our inspiration for our approach to $\gamma_{\mu, \nu}^\lambda$.

Theorem 1.5 (Rosas). *Given four sets of variables X, Y, T , and U ,*

$$HS_\lambda(XT, YU; XU, YT) = \sum \gamma_{\mu, \nu}^\lambda HS_\mu(X; Y) HS_\nu(T; U).$$

Lemma 1.6. *Let $\lambda \in H(k^2 + \ell^2; 2k\ell)$ be large and let X, Y, T , and U be sets of variables with cardinalities $|X| = |T| = k$ and $|Y| = |U| = \ell$. Then*

$$\prod_{\substack{x \in X \\ y \in Y}} (x + y)^{-1} \prod_{\substack{t \in T \\ u \in U}} (t + u)^{-1} HS_\lambda(XT, YU; XU, YT)$$

is a polynomial, symmetric in each of the four sets of variables. Hence, it can be expanded in terms of Schur functions. This expansion involves only typical μ and ν and equals is

$$\sum \gamma_{\mu, \nu}^\lambda S_{\alpha(\mu)}(X) S_{\beta(\mu)}(Y) S_{\alpha(\nu)}(T) S_{\beta(\nu)}(U).$$

Proof. By Theorem 1.5

$$HS_\lambda(XT, YU; XU, YT) = \sum \gamma_{\mu, \nu}^\lambda HS_\mu(X; Y) HS_\nu(T; U).$$

By Lemma 1.4 for each non-zero $\gamma_{\mu, \nu}^\lambda$ μ and ν are typical and so we may apply the Factorization Theorem, Theorem 0.1

$$HS_\mu(X; Y) = \prod (x + y) S_{\alpha(\mu)}(X) S_{\beta(\mu)}(Y)$$

and

$$HS_\nu(T; U) = \prod (t + u) S_{\alpha(\nu)}(T) S_{\beta(\nu)}(U).$$

The theorem now follows. \square

Definition 1.7. Given two finite sets of variables, the space of polynomials $f(X, Y)$ which are symmetric in each has an inner product with respect to which the $S_\mu(X) S_\nu(Y)$ are orthonormal. If $p(X)$ is defined to be the product of the elements of X and $p(Y)$ to be the product of the elements of Y , then the inner product extends to symmetric rational functions with denominator a power of $p(X)$ times a power of $p(Y)$ such that

$$\langle p(X)f(X, Y), p(X)g(X, Y) \rangle = \langle f(X, Y), g(X, Y) \rangle$$

and

$$\langle p(Y)f(X, Y), p(Y)g(X, Y) \rangle = \langle f(X, Y), g(X, Y) \rangle.$$

The inner product satisfies $\langle f(X, Y), g(X, Y) \rangle$ equals the coefficient of 1 in $(\Delta X)(\Delta Y)f(X, Y)g(X^{-1}, Y^{-1})$, where ΔX is the product of all $(1 - \frac{x_1}{x_2})$ over all distinct pairs of elements of X , and likewise for ΔY . This may also be expressed as a complex integral. Say $|X| = n$ and $|Y| = m$. Then $\langle f, g \rangle =$

$$\frac{1}{n!m!(2\pi i)^{n+m}} \oint_T (\Delta X)(\Delta Y)f(X, Y)g(X^{-1}, Y^{-1}) \frac{dx_1}{x_1} \wedge \dots \wedge \frac{dy_m}{y_m}$$

where T is the torus $|x_i| = |y_j| = 1$. Using this integral we may speak of the inner product of any two functions of X and Y .

Here is our main theorem.

Theorem 1.8. *If $\lambda \in H(k^2 + \ell^2; 2k\ell)$ is large, then m_λ equals*

$$\left\langle \prod (1 + x_i y_j^{-1})^{-1} (1 + x_i y_j^{-1})^{-1} HS_\lambda(XX^{-1}, YY^{-1}; XY^{-1}, YX^{-1}), 1 \right\rangle$$

where $|X| = k$ and $|Y| = \ell$.

Proof. By Lemma 1.6, noting that $(x+y)(x^{-1}+y^{-1})$ equals $(1+xy^{-1})(1+x^{-1}y)$, the inner product in the theorem equals

$$\begin{aligned} & \left\langle \sum \gamma_{\mu,\nu}^\lambda S_{\alpha(\mu)}(X) S_{\beta(\mu)}(Y) S_{\alpha(\nu)}(X^{-1}) S_{\beta(\nu)}(Y^{-1}), 1 \right\rangle \\ &= \sum \gamma_{\mu,\nu}^\lambda \langle S_{\alpha(\mu)}(X) S_{\beta(\mu)}(Y), S_{\alpha(\nu)}(X) S_{\beta(\nu)}(Y) \rangle \end{aligned}$$

The inner product is either 1 or 0, depending on whether $\alpha(\mu) = \alpha(\nu)$ and $\beta(\mu) = \beta(\nu)$. But this happens precisely when $\mu = \nu$ and so the sum is simply $\sum \gamma_{\mu,\mu}^\lambda$. \square

Notation 1.9. For $X = \{x_1, \dots, x_k\}$ and $Y = \{y_1, \dots, y_\ell\}$ we let $Z_0 = XX^{-1} \cup YY^{-1}$ and $Z_1 = XY^{-1} \cup YX^{-1}$. In this notation Theorem 1.8 can be stated as

$$m_\lambda = \left\langle \prod_{z_1 \in Z_1} (1 + z_1)^{-1} HS_\lambda(Z_0; Z_1), 1 \right\rangle.$$

Remark 1.10. In Corolary 21 of [3] we proved that $m_\lambda \leq \langle HS_\lambda(Z_0; Z_1), 1 \rangle$ for all λ

2 Integrals and Poincaré series

Recall that $T(k, \ell; a, b)$ is the sum $\sum m_\lambda S_{\alpha(\lambda)}(A) S_{\beta(\lambda)}(B)$, where the sum is over typical partitions and where A and B are sets of cardinality a and b , respectively. In order to compute this $T(k, \ell; a, b)$ from Theorem 1.8 we will need Cauchy's identity, see [8], I.4.3. For any sets of variables X and Y Cauchy's identity states

$$\sum_\lambda S_\lambda(X) S_\lambda(Y) = \prod_{\substack{x \in X \\ y \in Y}} (1 - xy)^{-1}.$$

We extend this slightly using the factorization theorem, Theorem 0.1.

$$\begin{aligned} & \sum_{\lambda \text{ typical}} HS_\lambda(A; B) S_{\alpha(\lambda)}(C) S_{\beta(\lambda)}(D) \\ &= \sum \prod (a + b) S_{\alpha(\lambda)}(A) S_{\beta(\lambda)}(B) S_{\alpha(\lambda)}(C) S_{\beta(\lambda)}(D) \\ &= \prod (a + b) \prod (1 - ac)^{-1} \prod (1 - bd)^{-1} \end{aligned} \tag{4}$$

where the a runs over A , the b runs over B , etc.

Theorem 2.1. $T(k, \ell; a, b)$ equals $(k!)^{-1}(\ell!)^{-1}(2\pi i)^{k+\ell}$ time the integral of

$$\prod (1 + z_1)^{-1} \prod (z_0 + z_1) \prod (1 - az_0)^{-1} \\ \prod (1 - bz_1)^{-1} \prod_{i \neq j} (1 - x_i x_j^{-1}) \prod_{i \neq j} (1 - y_i y_j^{-1}) \frac{dx_1}{x_1} \wedge \cdots \wedge \frac{dy_b}{y_b}$$

over the complex torus $|x_i| = 1$, $|y_i| = 1$, where the a, b, z_0, z_1 run over A, B, Z_0 and Z_1 , respectively.

Proof. By Theorem 1.8 if λ is typical $m_\lambda S_{\alpha(\lambda)}(A) S_{\beta(\lambda)}(B)$ equals the inner product with 1 of

$$\prod (1 + z_1)^{-1} H S_\lambda(Z_0; Z_1) S_{\alpha(\lambda)}(A) S_{\beta(\lambda)}(B).$$

Using equation (4) to sum this over all typical λ we get the inner product with 1 of

$$\prod (1 + z_1)^{-1} \prod (z_0 + z_1) \prod (1 - az_0)^{-1} \prod (1 - bz_1)^{-1}.$$

Interpreting the inner product as an integral as described in Definition 1.7 completes the proof. \square

In [13] Van Den Bergh studied integrals over the torus of the form

$$f(z_1, \dots, z_n) \prod_{i,j,k} (1 - z_1 z_j^{-1} x_k)^{-1}$$

where f is a degree 0 Laurent polynomial. Since the product $\prod (1 + x_i y_j^{-1})(1 + x_i y_j^{-1})$ divides evenly into $\prod (z_0 + z_1)$ this is the case here. Using his results we get this corollary.

Corollary 2.2. $T(k, \ell; a, b)$ is a rational function. The denominator can be written as a product of terms of the form $(1 - m)$ where m is a monic monomial of degree at most $k + \ell$, and of even degree in U . If a, b are so large that the integral in Theorem 2.1 has no poles at 0, then $T(k, \ell; a, b)$ satisfies the functional equation

$$T(t_1^{-1}, \dots, t_a^{-1}; u_1^{-1}, \dots, u_b^{-1}) = \\ (-1)^{(a-1)(k+\ell)+1} (t_1 \cdots t_a)^{k^2+\ell^2} (u_1 \cdots u_b)^{2k\ell} T(t_1, \dots, u_b)$$

Finally, recall $P'(k, \ell; a, b) = \sum m'_\lambda HS_\lambda(A; B)$, where $m'_\lambda = m_\lambda$ for large λ . The computation of this function as an integral is similar to the computation we just did for T , except that we need the following theorem of Berele and Remmel instead of equation (4). We leave the proof of Theorem 2.4 to the reader.

Theorem 2.3. (*Berele-Remmel [6]*) *Given four sets of variables A, B, C and D ,*

$$\sum_{\lambda} HS_{\lambda}(A; B) HS_{\lambda}(C; D) = \prod (1+ad) \prod (1+bc) \prod (1-ac)^{-1} \prod (1-bd)^{-1}.$$

Theorem 2.4. $P'(k, \ell; a, b)$ equals $(k!)^{-1}(\ell!)^{-1}(2\pi i)^{k+\ell}$ time the integral of

$$\prod (1+z_1)^{-1} \prod (1+z_0 b) \prod (1+z_1 a) \prod (1-z_0 a)^{-1} \\ \prod (1-z_1 b)^{-1} \prod_{i \neq j} (1-x_i x_j^{-1}) \prod_{i \neq j} (1-y_i y_j^{-1}) \frac{dx_1}{x_1} \wedge \cdots \wedge \frac{dy_b}{y_b}$$

This integral is not of the type discussed by Van Den Bergh and so we cannot easily derive an analogue of Corollary 2.2. It seems to us that P' might be less useful because of the potential presence of many terms $m'_\lambda HS_\lambda$ with m'_λ not equal to m_λ .

3 Computation of \bar{m}_λ

As mentioned in the introduction, there is another set of multiplicities we are interested in closely related to the m_λ . Before showing how to compute it for large or typical λ we first describe the analogues of problems 1, 3 and 4.

Problem 1b The sum $\sum (\chi^\lambda \otimes \chi^\lambda) \downarrow$ over $\lambda \in H(k, \ell; n+1)$ decomposes as a sum of irreducible characters $\sum \bar{m}_\lambda \chi^\lambda$ and we would like to compute the \bar{m}_λ .

Problem 3b The functions $\phi : M_{k,\ell}^n \rightarrow M_{k,\ell}$ which are polynomial in the entries and invariant under simultaneous conjugation from $PL(k, \ell)$ form an n -graded ring with Poincaré series equal to $\sum \bar{m}_\lambda S_\lambda(t_1, \dots, t_n)$

Problem 4b Referring to the notation of Problem 4a, let $R(k, \ell; n)$ be the algebra generated by the generic matrices A_1, \dots, A_n together with the trace ring $C(k, \ell; n)$. then $R(k, \ell; n)$ is an n -graded ring with Poincaré

series $\sum \bar{m}_\lambda S_\lambda(t_1, \dots, t_n)$. More generally, if $\bar{R}(k, \ell; n, m)$ is the algebra generated by $A_1, \dots, A_n, B_1, \dots, B_m$ and the supertrace ring $\bar{C}(k, \ell; n, m)$, then $\bar{R}(k, \ell; n, m)$ has Poincaré series

$$\bar{P}(k, \ell; n, m) = \sum \bar{m}_\lambda H S_\lambda(t_1, \dots, t_n; u_1, \dots, u_m).$$

We now turn to the computation of \bar{m}_λ . Given an S_n -character $\chi = \sum \alpha_\lambda \chi^\lambda$, we define $H(\chi)$ to be $\sum \alpha_\lambda H S_\lambda(Z_0; Z_1)$. This map respects addition and has two more properties we will need. First, H respects multiplication in the sense that

$$H(\chi_1 \hat{\otimes} \chi_2) = H(\chi_1) H(\chi_2). \quad (5)$$

This follows from [5]. The second property of H is a restatement of Theorem 1.8:

Lemma 3.1. *Let $\chi = \sum_{\lambda \vdash n} \alpha_\lambda \chi^\lambda$ be such that $\alpha_\lambda = 0$ unless λ is large. Then*

$$\langle \chi, \sum_{\mu \in H(k, \ell)} \chi^\mu \otimes \chi^\mu \rangle_{S_n} = \left\langle \prod_{z_1 \in Z_1} (1 + z_1)^{-1} H(\chi), 1 \right\rangle.$$

Proof. Each side of the equation equals $\sum \alpha_\lambda m_\lambda$. □

Theorem 3.2. *For each large λ the multiplicity \bar{m}_λ equals*

$$\left\langle \sum_{z \in Z_0 \cup Z_1} z \prod_{z_1 \in Z_1} (1 + z_1)^{-1} H S_\lambda(Z_0; Z_1), 1 \right\rangle.$$

Proof. By definition, $\bar{m}_\lambda = \langle \chi^\lambda, \sum (\chi^\mu \otimes \chi^\mu) \downarrow \rangle_{S_n}$, summed over $\mu \in H(k, \ell; n+1)$. By Froebenius reciprocity this equals $\langle \chi^\lambda \uparrow, \sum \chi^\mu \otimes \chi^\mu \rangle_{S_{n+1}}$ and $\chi^\lambda \uparrow$ equals $\chi^{[1]} \hat{\otimes} \chi^\lambda$. Applying the previous lemma we get

$$\bar{m}_\lambda = \left\langle \prod_{z_1 \in Z_1} (1 + z_1)^{-1} H(\chi^{[1]} \hat{\otimes} \chi^\lambda), 1 \right\rangle.$$

By (5)

$$\begin{aligned} H(\chi^{[1]} \hat{\otimes} \chi^\lambda) &= H(\chi^{[1]}) H(\chi^\lambda) \\ &= H S_{[1]}(Z_0; Z_1) H S_\lambda(Z_0; Z_1) \\ &= \left(\sum_{z \in Z_0 \cup Z_1} z \right) H S_\lambda(Z_0; Z_1). \end{aligned}$$

The theorem now follows. □

Theorem 3.2 easily implies analogues of Theorems 2.1 and 2.4, and the former implies an analogue of Corollary 2.2. The Poincaré series we study are $\bar{T}(k, \ell; a, b)$ which equals the sum $\sum \bar{m}_\lambda S_{\alpha(\lambda)}(T) S_{\beta(\lambda)}(U)$ summed over typical λ ; and $\bar{P}'(k, \ell; a, b)$ which equals the sum $\sum \bar{m}'_\lambda HS_\lambda(T; U)$, where \bar{m}'_λ is the inner product in Theorem 3.2. Here are the results.

Theorem 3.3. $\bar{T}(k, \ell; a, b)$ equals $(k!)^{-1}(\ell!)^{-1}(2\pi i)^{k+\ell}$ time the integral of

$$\sum_{z \in Z_0 \cup Z_1} z \prod (1 + z_1)^{-1} \prod (z_0 + z_1) \prod (1 - az_0)^{-1} \\ \prod (1 - bz_1)^{-1} \prod_{i \neq j} (1 - x_i x_j^{-1}) \prod_{i \neq j} (1 - y_i y_j^{-1}) \frac{dx_1}{x_1} \wedge \cdots \wedge \frac{dy_b}{y_b}$$

over the complex torus $|x_i| = 1$, $|y_i| = 1$, where the a, b, z_0, z_1 run over A, B, Z_0 and Z_1 , respectively.

Corollary 3.4. $\bar{T}(k, \ell; a, b)$ is a rational function. The denominator can be written as a product of terms of the form $(1 - m)$ where m is a monomial of degree at most $k + \ell$, and of even degree in U . If a, b are so large that the integral in Theorem 3.3 has no poles at 0, then $T(k, \ell; a, b)$ satisfies the functional equation

$$T(t_1^{-1}, \dots, t_a^{-1}; u_1^{-1}, \dots, u_b^{-1}) = \\ (-1)^{(a-1)(k+\ell)+1} (t_1 \cdots t_a)^{k^2+\ell^2} (u_1 \cdots u_b)^{2k\ell} T(t_1, \dots, u_b)$$

Theorem 3.5. $\bar{P}'(k, \ell; a, b)$ equals $(k!)^{-1}(\ell!)^{-1}(2\pi i)^{k+\ell}$ time the integral of

$$\sum_{z \in Z_0 \cup Z_1} z \prod (1 + z_1)^{-1} \prod (1 + z_0 b) \prod (1 + z_1 a) \prod (1 - z_0 a)^{-1} \\ \prod (1 - z_1 b)^{-1} \prod_{i \neq j} (1 - x_i x_j^{-1}) \prod_{i \neq j} (1 - y_i y_j^{-1}) \frac{dx_1}{x_1} \wedge \cdots \wedge \frac{dy_b}{y_b}$$

4 Examples

4.1 Case of λ one row or one column

We first compute m_λ for $\lambda = (n)$ and $\lambda = (1^n)$ and k, ℓ arbitrary. Such λ will be neither large nor typical and we compute it directly from the inner product definition $m_\lambda = \sum \chi^\lambda \otimes \chi^\lambda$ summed over $\lambda \in H(k, \ell; n)$.

Lemma 4.1. *With γ as in Notation 1.1, $\gamma_{\mu,\nu}^{(n)} = \delta_{\mu,\nu}$ and $\gamma_{\mu,\nu}^{(1^n)} = \delta_{\mu,\nu'}$*

Proof. Referring to I.7 of [8], $\chi^{(n)} \otimes \chi^\mu = \chi^\mu$ for any μ and so $\gamma_{(n),\nu}^\mu = \delta_{\mu,\nu}$. But, considered as a function of λ, μ, ν , $\gamma_{\mu,\nu}^\lambda$ is symmetric and so $\gamma_{\mu,\nu}^{(n)} = \delta_{\mu,\nu}$, as claimed. The second statement follows similarly using the identity $\chi^{(1^n)} \otimes \chi^\mu = \chi^{\mu'}$ from example 2 in section I.7 of [8]. \square

Theorem 4.2. *$m_{(n)}$ equals the number of partitions in $H(k, \ell; n)$ and $m_{(1^n)}$ equals the number of self-conjugate partitions in $H(k, \ell; n)$. In particular, $m_{(n)} \neq m_{(1^n)}$ and so $m_\lambda \chi^\lambda$ is not symmetric under conjugation.*

Proof. By definition $m_{(n)} = \sum \gamma_{\mu,\mu}^{(n)}$, summed over $\mu \in H(k, \ell; n)$; and by lemma 4.1 each $\gamma_{\mu,\mu}^{(n)}$ equals 1 and so the sum is $|H(k, \ell; n)|$. The case of $m_{(1^n)}$ is similar, with $\gamma_{\mu,\mu}^{(1^n)}$ equaling 1 if μ is self-conjugate and 0 otherwise. \square

Corollary 4.3. *Let $f(x) = P(1, 1; 0, 1)$, $g(x) = P(2, 2; 1, 0)$, and let $h(x)$ equal either $f(x)$ or $g(x)$. Then $h(x)$ does not satisfy a functional equation $h(x^{-1}) = \pm x^a h(x)$.*

Proof. A partition $\lambda \in H(1, 1)$ is self-conjugate if and only if $\lambda = [0]$ or $\lambda = [a + 1, 1^a]$ for some $a \geq 0$. Hence,

$$f(x) = 1 + \sum x^{2a+1} = 1 + \frac{x}{1-x^2} = \frac{1+x-x^2}{1-x^2}.$$

For $g(x)$, note that if $\lambda \in H(2, 2)$, either λ is typical or $\lambda \in H(1, 1)$. If λ is typical, then $|\lambda| = 4 + |\alpha(\lambda)| + |\beta(\lambda)|$, and since $\alpha(\lambda)$ and $\beta(\lambda)$ are partitions of height at most 2, it follows that

$$\sum_{\lambda \in H'(2,2)} t^{|\lambda|} = \frac{t^4}{[(1-t)(1-t^2)]^2}.$$

If $\lambda \in H(1, 1)$, then either $\lambda = [0]$ or λ is typical. It follows that

$$\sum_{\lambda \in H(1,1)} t^{|\lambda|} = 1 + \frac{x}{(1-x)^2}.$$

Adding, we get

$$\begin{aligned} g(x) &= 1 + \frac{x}{(1-x)^2} + \frac{x^4}{(1-x)^2(1-x^2)^2} \\ &= \frac{1-x+x^2+x^3-x^4}{(1-x)^2(1-x^2)^2}. \end{aligned}$$

It is now easy to see that neither $f(x)$ nor $g(x)$ satisfy a functional equation of the indicated type. \square

4.2 The case of $(k, \ell) = (1, 1)$

We now turn to the $k = \ell = 1$ case. This case was computed directly by Regev in [10] and Remmel in [11], so this computation is mostly a check on Theorem 2.1. In this case $X = \{x\}$ and $Y = \{y\}$, and so $Z_0 = \{1, 1\}$ and $Z_1 = \{\frac{x}{y}, \frac{y}{x}\}$. Hence, after canceling the $(1 + \frac{x}{y})(1 + \frac{y}{x})$ in the denominator with one of the two in the numerator, the integrand in Theorem 2.1 equals

$$(1 + \frac{x}{y})(1 + \frac{y}{x}) \prod_i (1 - t_i)^{-2} \prod_j (1 - \frac{x}{y}u_j)^{-1} (1 - \frac{y}{x}u_j)^{-1} \quad (6)$$

Since $(k^2 + \ell^2; 2k\ell) = (2, 2)$, we will take the case of two t 's and two u 's. The first integral is by $\frac{dx}{x}$ and the poles inside the circle are at $x = yu_1$ and $x = yu_2$. Pulling out the factor of $(1 - t_1)^2(1 - t_2)^2$, the pole at yu_1 equals:

$$\frac{(1 + u_1^{-1})(1 + u_1)}{(1 - u_1u_2)(1 - u_2u_1^{-1})(1 - u_1^2)}$$

which simplifies to

$$\frac{(1 + u_1)}{(1 - u_1u_2)(u_1 - u_2)(1 - u_1)}.$$

Integrating by $\frac{dy}{y}$ leaves this as it is. Similarly, if we use the pole at $x = yu_2$ we simply switch u_1 and u_2 yielding

$$\frac{(1 + u_2)}{(1 - u_1u_2)(u_2 - u_1)(1 - u_2)}.$$

Adding these two fractions and multiplying back the $(1 - t_1)^2(1 - t_2)^2$ we factored out we get $T(1, 1; 2, 2)$.

Theorem 4.4. $T(1, 1; 2, 2) =$

$$\frac{2}{(1 - t_1)^2(1 - t_2)^2(1 - u_1)(1 - u_2)(1 - u_1u_2)}.$$

It follows from the identities $\sum S_\lambda(u_1, u_2) = (1 - u_1)^{-1}(1 - u_2)^{-1}(1 - u_1u_2)^{-1}$ and $\sum (\lambda_1 - \lambda_2 + 1)S_\lambda(t_1, t_2) = (1 - t_1)^{-2}(1 - t_2)^{-2}$ that if λ and typical with $\alpha(\lambda) = (\alpha_1, \alpha_2)$ and $\beta(\lambda) = (\beta_1, \beta_2)$, then $m_\lambda = 2(\alpha_1 - \alpha_2 + 1)$, in agreement with [11].

In order to find m_λ for smaller λ , note that every $\lambda \in H(1, 1)$ is typical, except for $\lambda = [0]$, and so $m_\lambda = m'_\lambda$ for all λ except for $m_{[0]} = 1$ and

$m'_{[0]} = 0$. Hence, we may use $P'(1, 1; 1, 1)$ to compute m_λ in the $(1, 1)$ -hook. By Theorem 2.4, we compute $P'(1, 1; 1, 1)$ by integrating

$$\frac{(1+u)^2(1+\frac{x}{y}t)(1+\frac{y}{x}t)}{(1+\frac{x}{y})(1+\frac{y}{x})(1-t)^2(1-\frac{x}{y}u)(1-\frac{y}{x}u)} \frac{dx}{x} \wedge \frac{dy}{y} \quad (7)$$

There are two poles: One at $x = yu$ and one at $x = -y$. The former has residue

$$\frac{(1+u)^2(1+ut)(1+\frac{t}{u})}{(1+u)(1+u^{-1})(1-t)^2(1-u^2)}$$

which equals

$$\frac{(1+ut)((t+u))}{(1-t)^2(1-u^2)}$$

times $\frac{dy}{y}$. The pole in (7) at $x = -y$ is of order two, so in order to compute the residue we must first multiply by $(x+y)^2$, then take the partial derivative with respect to x , and finally substitute $-y$ for x . The computation is a bit long and the result is 0.

Theorem 4.5. $P'(1, 1; 1, 1) = (1+ut)(t+u)(1-t)^{-2}(1-u^2)^{-1}$.

It follows that if $\lambda = (a+1, 1^b)$, then m_λ equals $a+1$ if b is even and a if b is odd.

A similar analysis can be carried out for \bar{m}_λ . We leave the proof to the reader.

Theorem 4.6. $\bar{P}'(1, 1; 1, 1) = 1 + 2(t+u)(1+tu)(1-t)^{-2}(1-u)^{-1}$ which implies that $\bar{m}_{[a+1, 1^b]}$ equals $4a+2$ if $b > 0$ and $\bar{m}_{[a+1]} = 2a+2$. Also,

$$\begin{aligned} \bar{P}'(1, 1; 2, 2) &= \frac{2(3+u_1+u_2-u_1u_2)}{(1-t_1)^2(1-t_2)^2(1-u_2)(1-u_2)(1-u_1u_2)} \\ &= \frac{1}{(1-t_1)^2(1-t_2)^2} \left(\frac{8}{(1-u_2)(1-u_2)(1-u_1u_2)} - \frac{2}{1-u_1u_2} \right) \end{aligned}$$

It follows that if λ is typical with $\alpha(\lambda) = (\alpha_1, \alpha_2)$ and $\beta(\lambda) = (\beta_1, \beta_2)$, then $\bar{m}_\lambda = 8(\alpha_1 - \alpha_2 + 1)$ unless $\beta_1 = \beta_2$ in which case $\bar{m}_\lambda = 6(\alpha_1 - \alpha_2 + 1)$.

4.3 The case of $(k, \ell) = (2, 1)$

We conclude with a peek into the unknown, thanks to a Maple computation:

$$T(2, 1; 1, 0) = \frac{372 + 801t + 835t^2 + 515t^3 + 213t^4 + 35t^5 + t^6}{(1-t)^2(1-t^2)}, \quad (8)$$

$$T(2, 1; 0, 1) = (1 - u)^{-1}(372 + 780u + 1083u^2 + 1193u^3 + 1034u^4 + 754u^5 + 513u^6 + 319u^7 + 158u^8 + 54u^9 + 11u^{10} + u^{11}), \quad (9)$$

$$\bar{T}(2, 1; 1, 0) = \frac{2697 + 6346t + 6641t^2 + 4449t^3 + 1981t^4 + 503t^5 + 50t^6 + t^7}{(1 - t)^2(1 - t^2)}, \quad (10)$$

and

$$\begin{aligned} \bar{T}(2, 1; 0, 1) = (1 - u)^{-1}(2697 + 6249u + 8817u^2 + 9587u^3 + 8706u^4 \\ + 6890u^5 + 4877u^6 + 3107u^7 + 1744u^8 + 820u^9 + 301u^{10} \\ + 79u^{11} + 13u^{12} + u^{13}) \end{aligned} \quad (11)$$

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